Exercise 42

A sphere with radius 1 m has temperature 15°C. It lies inside a concentric sphere with radius 2 m and temperature 25°C. The temperature $T(r)$ at a distance r from the common center of the spheres satisfies the differential equation

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0$$

If we let $S = dT/dr$, then S satisfies a first-order differential equation. Solve it to find an expression for the temperature $T(r)$ between the spheres.

Solution

$$S = \frac{dT}{dr} \quad \rightarrow \quad \frac{dS}{dr} = \frac{d}{dr}\frac{dT}{dr} = \frac{d^2T}{dr^2}$$

Substituting these expressions into the differential equation gives

$$\begin{aligned} \frac{dS}{dr} + \frac{2}{r}S &= 0 \\ \frac{dS}{dr} &= -\frac{2}{r}S. \end{aligned}$$

This is a separable differential equation, so we can solve for $S(r)$ by bringing the terms with S to the left and the constants and terms with r to the right and then integrating both sides.

$$\begin{aligned} dS &= -\frac{2}{r}S\,dr \\ \frac{dS}{S} &= -\frac{2}{r}\,dr \\ \int \frac{dS}{S} &= \int -\frac{2}{r}\,dr \\ \ln|S| &= -2\ln|r| + C \\ \ln|S| &= \ln|r|^{-2} + C \\ e^{\ln|S|} &= e^{\ln|r|^{-2}+C} \\ |S| &= e^C|r|^{-2} \\ S(r) &= \pm e^C|r|^{-2} \end{aligned}$$

The absolute value around r can be dropped since $r > 0$. Also, let $C_1 = \pm e^C$.

$$\begin{aligned} S(r) &= C_1 r^{-2} \\ \frac{dT}{dr} &= C_1 r^{-2} \\ dT &= C_1 r^{-2}\,dr \\ \int dT &= \int C_1 r^{-2}\,dr \\ T(r) &= C_1 \frac{1}{-1} r^{-1} + C_2 \end{aligned}$$

$$T(r) = -C_1 \frac{1}{r} + C_2$$

To solve for the integration constants, we need to use the boundary conditions given in the problem. We are told that the temperature at the surface of the smaller sphere is 15°C and that the temperature at the surface of the larger sphere is 25°C. Therefore, $T(1) = 15$ and $T(2) = 25$.

$$T(1) = -C_1 \frac{1}{1} + C_2 = 15$$
$$T(2) = -C_1 \frac{1}{2} + C_2 = 25$$

Solving this system of equations gives $C_1 = 20$ and $C_2 = 35$. Thus, the equation for the temperature is as follows.

$$T(r) = -20 \frac{1}{r} + 35$$

The function is plotted below versus r from 1 to 2.

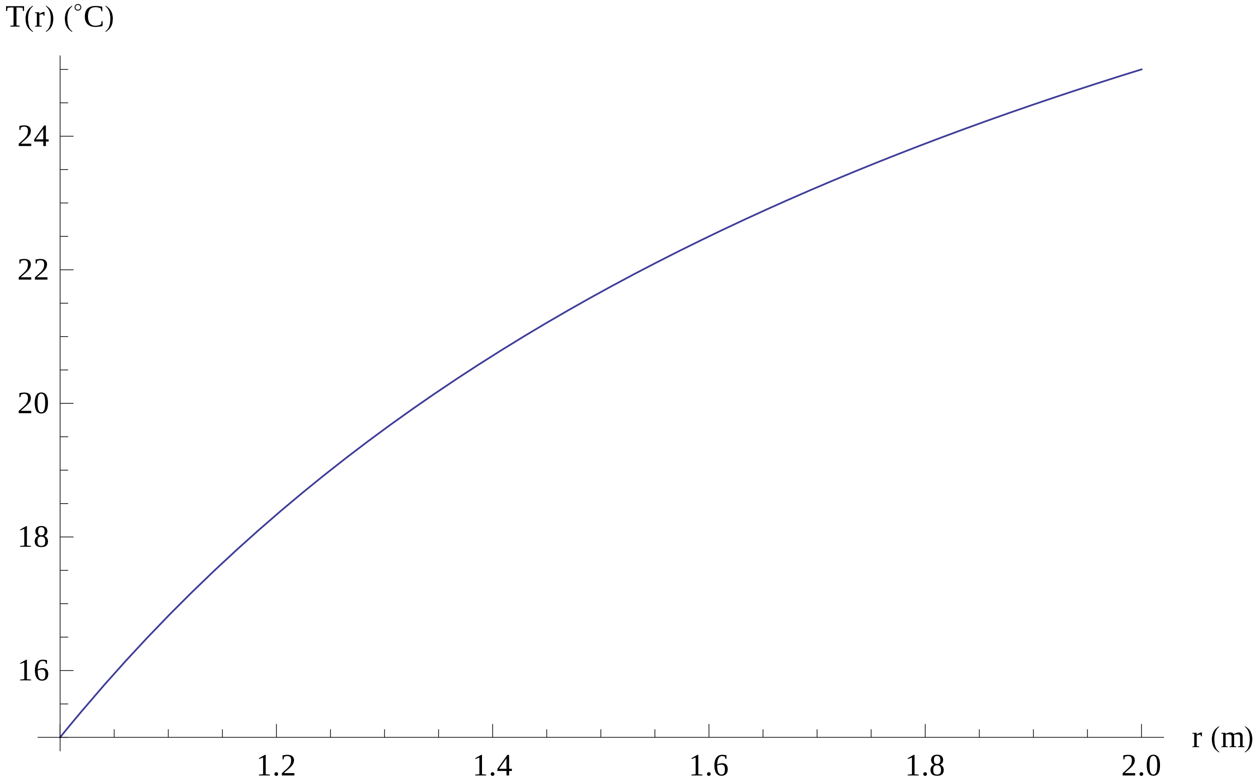

Figure 1: Plot of $T(r)$ vs. r.